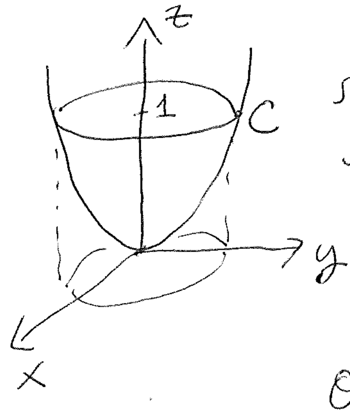
MATH 2020 B

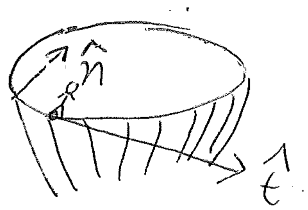
April 24

One more example on Stokes' thm.

e.g. Let S be the elliptical paraboloid $z = x^2 + 4y^2$ lying beneath $z = 1$. Find the flux of $\nabla \times \vec{F}$, $\vec{F} = y\hat{i} - xz\hat{j} + xz^2\hat{k}$, across S. Take $\hat{n}$ to be the inner normal.



According to the "left hand rule", the orientation of C is anticlockwise as viewed from above.



The boundary of S, C, is described by

$$\theta \mapsto (\cos\theta, \tfrac{1}{2}\sin\theta, 1), \quad \theta \in [0, 2\pi]$$
$$= \cos\theta\,\hat{i} + \tfrac{1}{2}\sin\theta\,\hat{j} + \hat{k}.$$

Using Stokes' thm,

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} (y\hat{i} - xz\hat{j} + xz^2\hat{k}) \cdot (-\sin\theta\,\hat{i} + \tfrac{1}{2}\cos\theta\,\hat{j} + 0\,\hat{k})$$

$$= \int_0^{2\pi} y(-\sin\theta) - xz\tfrac{1}{2}\cos\theta \, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\sin^2\theta - \frac{1}{2}\cos^2\theta \, d\theta$$

$$= -\frac{1}{2} \times 2\pi = -\pi \quad \#$$

We present two consequences of Stokes' thm.

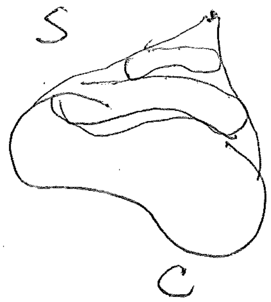
Consequence (I)

Theorem Let $\vec{F}$ be a smooth v.f. in a simply-connected region Ω in space. Then $\vec{F}$ is conservative if and only if it satisfies the component test.

Pf. $\Rightarrow$) done already (true for any region)

$\Leftarrow$) By simply-connected condition, any simple closed curve in Ω can be deformed continuously to a point in Ω. The deformation forms a surface S which is bdd by C.

By Stokes' thm



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, d\sigma,$$

But

$$\nabla\times\vec{F} = (P_y - N_z)\,\hat{i} - (P_x - M_z)\,\hat{j} + (N_x - M_y)\,\hat{k}$$
$$= 0 \quad \text{by compnt test}$$

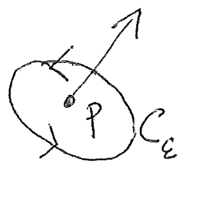
so,

$$\oint_C \vec{F}\cdot d\vec{r} = 0,$$

ie $\vec{F}$ satisfies the loop property, so it is conservative. #

Consequence (II) the meaning of $\nabla\times\vec{F}$.

Fix $P(x,y,z)$ in Ω and take a unit vector $\hat{\xi}$.
Let D_ε be a disk of radius ε around P whose $\hat{n}$ is $\hat{\xi}$.
Regarding D_ε as an oriented surface,



$$\oint_{C_\varepsilon} \vec{F}\cdot d\vec{r} = \iint_{D_\varepsilon} \nabla\times\vec{F}\cdot\hat{n}\, d\sigma = \iint_{D_\varepsilon} \nabla\times\vec{F}\cdot\hat{\xi}\, d\sigma$$

$\therefore \quad \nabla\times\vec{F}(P)\cdot\hat{\xi} = \lim_{\varepsilon\to 0} \frac{1}{|D_\varepsilon|} \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r}.$

That's, $\nabla\times\vec{F}$ is the vector which $\nabla\times\vec{F}\cdot\hat{\xi}$ gives the localized circulation of $\vec{F}$ around P along the direction $\hat{\xi}$.

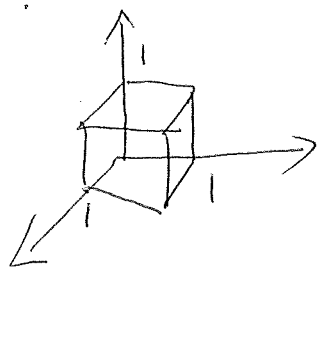
The divergence thm (Gauss' thm) is the 3-dim version of the Green's theorem.

<u>Theorem</u> Let $\vec{F}$ be a smooth v.f. in a region Ω in space.

$$\iiint_\Omega \operatorname{div}\vec{F}\,dV = \oiint_S \vec{F}\cdot\hat{n}\,d\sigma$$

where S is the boundary of Ω and $\hat{n}$ the outward pointing normal.

e.g. Find the flux of $xy\hat{i} + yz\hat{j} + xz\hat{k}$ through the surface of the cube $x, y, z = 1$ in the first octant.



Instead of performing 6 surface integration, we use Gauss' thm

$$\oiint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_\Omega \operatorname{div}\vec{F}\,dV$$

$$= \iiint_\Omega \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz)$$

$$= \int_0^1\int_0^1\int_0^1 (y+z+x)\,dx\,dy\,dx$$

$$= 3\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz \quad \text{(by symmetry)}$$

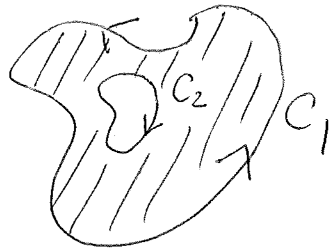
$$= 3/2 \ \#$$

Using the technique of cutting-up, Green's, Stokes',
and Gauss' theorems can be generalized.

Green's thm Let D be a region bdd by $C_1, \dots C_n$ where
C_1 is the outer one. Then

$$\iint_D (N_x - M_y)\, dA = \sum_j \oint_{C_j} \vec{F} \cdot d\vec{r}.$$

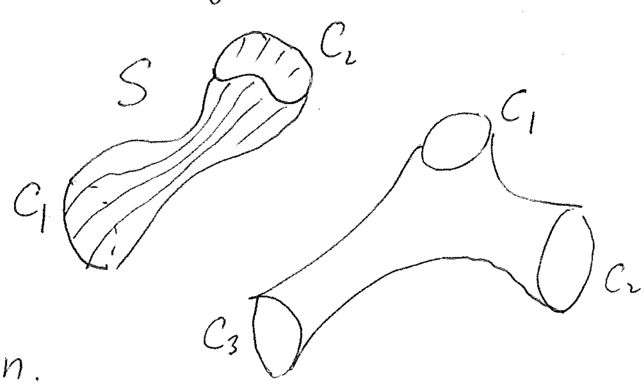
Here C_1 taking anticlockwise direction and $C_2, \dots C_n$ in clockwise direction.



Stokes' thm Let S be bdd by $C_1, \dots C_n$. Then

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \sum_j \oint_{C_j} \vec{F} \cdot d\vec{r},$$

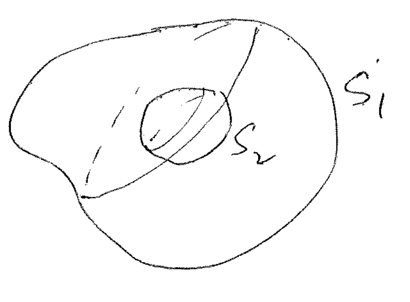
when C_j has orientation according to the "left hand rule".



Div. thm Let Ω be bdd by $S_1, \dots S_n$.

$$\iiint_\Omega \operatorname{div} \vec{F}\, dV = \sum_j \oiint_{S_j} \vec{F} \cdot \hat{n}\, d\sigma,$$

when $\hat{n}$ are outer normal.



A Consequence of div. theorem.

A point charge generates an electric field:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\vec{F}, \quad \vec{F} = \frac{1}{|\vec{r}|^3}\vec{r}$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\hat{i}+y\hat{j}+z\hat{k})$$

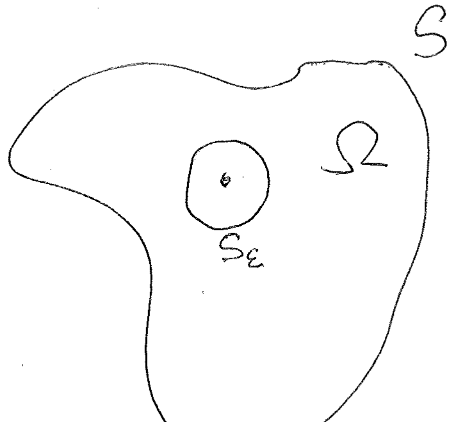
where the origin $\vec{O}$ is the location of the charge.

Gauss' law For any surface S enclosing the origin, the flux of $\vec{E}$ across S is the same equal to q/ε_0, ie,

$$\iint_S \vec{E}\cdot\hat{n}\, d\sigma = q/\varepsilon_0.$$

Pf: Let B_ε be a small ball $B_\varepsilon \subset \Omega$ and let $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$.
Ω_ε is region bdd by S and S_ε (the boundary of B_ε)

$$\therefore \iint_S \vec{F}\cdot\hat{n} + \iint_{S_\varepsilon} \vec{F}\cdot\hat{n} = \iiint_{\Omega_\varepsilon} \operatorname{div} F\, dV$$



(can't apply Gauss thm to Ω as $\vec{F}$ is not defined at $\vec{O}$)

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}},$$

$$\text{1st term} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - x \times \frac{3}{2}\frac{2x}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}.$$

Similarly,

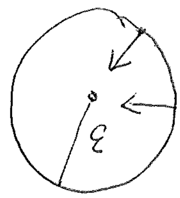
$$\frac{\partial}{\partial y}\frac{y}{(\quad)^{3/2}} = \frac{x^2-2y^2+z^2}{(x^2+y^2+z^2)^{5/2}},$$

$$\frac{\partial}{\partial z}\frac{z}{(\quad)^{3/2}} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}.$$

$\therefore \quad \operatorname{div}\vec{F} = 0.$

Thus,

$$\oiint_S \vec{F}\cdot\hat{n}\,d\sigma = -\oiint_{S_\varepsilon} \vec{F}\cdot\hat{n}\,d\sigma.$$



On S_ε, $\hat{n} = \frac{1}{\varepsilon}(-x\hat{i}-y\hat{j}-z\hat{k})$

$$\vec{F} = \frac{1}{\varepsilon^3}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\vec{F}\cdot\hat{n} = \frac{1}{\varepsilon^4}(x^2+y^2+z^2) = \frac{-1}{\varepsilon^2}$$

$$\oiint_{S_\varepsilon} \vec{F}\cdot\hat{n}\,d\sigma = -\frac{1}{\varepsilon^2}\iint_{S_\varepsilon} d\sigma = -\frac{1}{\varepsilon^2}4\pi\varepsilon^2 = -4\pi$$

We conclude

$$\oiint_S \vec{F}\cdot\hat{n}\,d\sigma = 4\pi$$

$$\therefore \quad \oiint_S \vec{E}\cdot\hat{n}\,d\sigma = \frac{q}{4\pi\varepsilon_0}\oiint_S \vec{F}\cdot n\,d\sigma$$

$$= \frac{q}{4\pi\varepsilon_0}\times 4\pi$$

$$= q/\varepsilon_0 \quad \#$$